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Cycle bases from orderings and coverings

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Abstract

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We use notions of 'ordering' and 'covering' to define several classes (some new, some old) of cycle bases for graphs. Each class is characterized in terms of both its structure and its relationship to the other classes. For each class we also characterize (both constructively and by excluded subgraphs) those graphs for which every cycle basis is in the class.

1. Introduction

All graph theory terminology, unless otherwise stated, is consistent with [1]. The graphs we consider have no loops but may have multiple edges. To simplify our definitions and results, we consider only 2-connected graphs. (All results can be generalized quite easily.)

The standard definition of a cycle in a graph is a minimal subgraph such that every node has degree 2. However, we use the term *cycle* to refer to the edge sets of such subgraphs.

To each cycle C in a graph G, associate an incidence vector x where $x_e = 1$ if $e \in C$ and $x_e = 0$, otherwise. The vector space over GF(2) generated by these vectors is called the *cycle space* of G. A collection of cycles whose incidence vectors form a basis for the cycle space of a graph is called a *cycle basis.*

In this paper we present several classes of cycle bases defined in terms of 'orderings' and 'coverings'. In the remainder of this section we introduce the classes of cycle bases that we study, put them into historical perspective, and then outline our results.

Suppose $\mathcal{P} = \{C_1, \ldots, C_d)$ is a cycle basis for a 2-connected graph G. For an ordering π of $\mathcal{P}$, let G_i be the graph induced by $C_{\pi(1)} \cup \cdots \cup C_{\pi(i)}$. An ordering

π is called *fundamental* if each G_i is a proper subgraph of G_{i+1}. π is called *block expanding* if each G_i is 2-connected. $\mathscr{P}$ is called *fundamental* or *F* (*block expanding* or *BE*) if it has an ordering that is fundamental (block expanding). $\mathscr{P}$ is called *strictly fundamental* or *SF* (*strictly block expanding* or *SBE*) if every ordering is fundamental (block expanding). $\mathscr{P}$ is called an *ear decomposition* or *ED* cycle basis if it has an ordering that is both fundamental and block expanding.

Finally, define a hierarchy of classes of cycle bases, one class for each positive integer, as follows: Let us say that an edge e in a graph G is *covered k times* by a collection of cycles $\mathscr{P}$ if e is contained in at least k distinct cycles of $\mathscr{P}$. A cycle basis $\mathscr{P}$ that covers every edge of a graph k times is called a *k-covering* or *k-C cycle basis.*

Before stating our results, we make a few historical remarks. The SF cycle bases are the well-known class arising from spanning trees (see Section 2) which were introduced by Kirchhoff in 1847 [3]. The name was coined by Whitney in a paper [6] where he also introduced the more general notion of F cycle bases. We have named the ED cycle bases for the closely related notion (also introduced by Whitney [5]) of ear decompositions (see Section 2).

The paper is organized as follows: In Section 2 we characterize the SF, SBE, ED, F and BE cycle bases. In Section 3 we show that $\{$SF cycle bases$\} \subseteq \{$ED cycle bases$\} \subseteq \{F$ cycle bases$\} \subseteq \{$BE cycle bases$\}$. We also show how the SBE and k-C cycle bases relate to the others (see Fig. 1).

Let G_{SF}, G_{SBE}, G_{ED}, G_F and G_{BE} denote the classes of graphs such that every cycle basis is SF, SBE, ED, F and BE, respectively. In Sections 4, 5 and 6, we characterize these classes of graphs both constructively and in terms of excluded subgraphs. We find that $G_{\mathrm{SF}} \subseteq G_{\mathrm{ED}} = G_F \subseteq G_{\mathrm{BE}} = \{$all 2-connected graphs$\}$. We also show that a 2-connected graph has a cycle basis that is not ED iff it has a 2-C cycle basis.

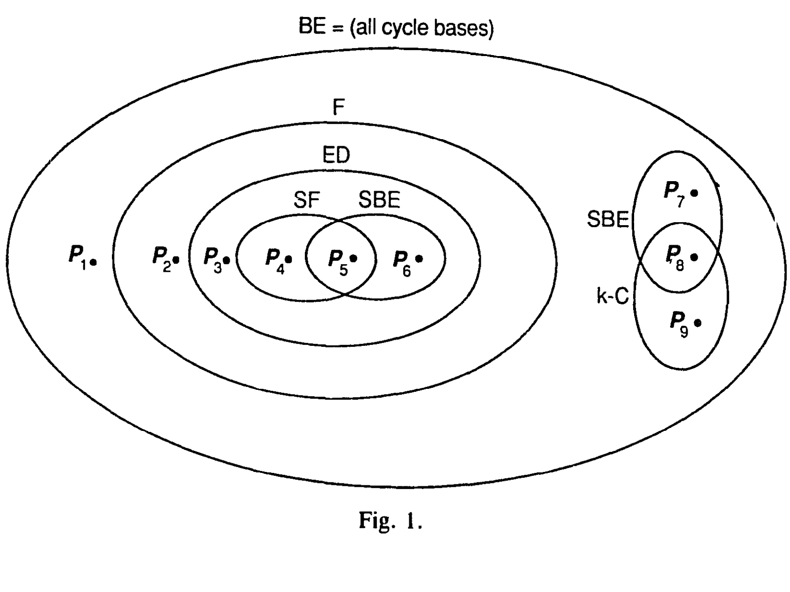


Fig. 1.

In Section 7 we show that k-C cycle bases exist for all $k \geq 1$.

2. Characterizing the SF, SBE, ED, F and BE cycle bases

In this section we characterize the five classes of cycle bases that arise from our two notions of ordering. The characterizations of the SF, SBE and BE cycle bases are quite simple and we present them first. We next present the characterization of the F cycle bases with which the ED cycle bases are easily characterized.

Given a 2-connected graph G, let $T = (V, E')$ be a maximal spanning tree for G and let $\mathcal{P}$ consist of the unique cycle in $e \cup E'$ for each $e \in E \setminus E'$. Then $\mathcal{P}$ is called a *Kirchhoff* cycle basis. (Kirchhoff originally made this definition in 1847 [3].)

Proposition 2.1. *A cycle basis $\mathcal{P}$ is SF iff it is Kirchhoff.*

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) Suppose $\mathcal{P}$ is an SF cycle basis for $G = (V, E)$. Every cycle in $\mathcal{P}$ contains at least one edge that is contained in no other cycle. Let G' be obtained from G by deleting one such edge from each cycle in $\mathcal{P}$. G' can contain no cycles since $\mathcal{P}$ is a cycle basis. It is well known that for G connected, $|\mathcal{P}| = |E| - |V| + 1$. Hence G' has $|V| - 1$ edges and is a spanning tree. □

Proposition 2.2. *A cycle basis $\mathcal{P}$ is SBE iff the graph induced by every pair of cycles of $\mathcal{P}$ is 2-connected (i.e., every pair of cycles has at least two common nodes).*

Proof. ($\Rightarrow$) If there exist two cycles that do not have this property, then any ordering in which these two occur first and second is not SBE.

($\Leftarrow$) Obvious. □

In order to characterize the BE cycle bases we use the following proposition due to Thomassen [4].

Proposition 2.3. *Let $\mathcal{P}$ be a cycle basis for a 2-connected graph $G = (V, E)$. Then for any partition E_1, E_2 of E, there exists a cycle $C \in \mathcal{P}$ such that $C \cap E_1 \neq \emptyset$ and $C \cap E_2 \neq \emptyset$.*

Proposition 2.4. *Every cycle basis $\mathcal{P}$ for a 2-connected graph G is BE.*

Proof. Let $G' = (V', E')$ be a graph with one node for each cycle in $\mathcal{P}$ and $(v_1, v_2) \in E'$ iff the cycles in $\mathcal{P}$ corresponding to v_1 and v_2 have a common edge.

We first show that G' is connected. If not, then there exists a partition of the edges in G such that no cycle in $\mathcal{P}$ contains an edge from each set. This implies by Proposition 2.3 that G is not 2-connected. Contradiction.

To construct a block enlarging ordering of $\mathcal{P}$, let T' be a spanning tree of G'. For $i := |\mathcal{P}|$ to 1, let $C_{\pi(i)}$ correspond to a degree 1 node of the subtree of T' induced by the cycles of $\mathcal{P}$ not yet ordered. Then π is a block enlarging ordering of $\mathcal{P}$. □

For $G=(V, E)$ and $E' \subseteq E$, let $G(E')$ denote the subgraph of G induced by the edges in E'. Define the intersection of two graphs, $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$, to be the graph $G=(V_1 \cap V_2, E_1 \cap E_2)$.

Proposition 2.5. *Let $\mathcal{P}=\{C_1, \ldots, C_d\}$ be a cycle basis for a graph $G=(V, E)$. Then $\mathcal{P}$ is an F cycle basis relative to the ordering $C_1, \ldots, C_d$ iff the intersection of $G(C_j)$ with every component of $G(C_1 \cup \cdots \cup C_{j-1})$, for $2 \leqslant j \leqslant d$, is a path (possibly of length zero) or empty.*

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) Let $G_{j-1}=(V_{j-1}, E_{j-1})=G(C_1 \cup \cdots \cup C_{j-1})$ and $G_j=(V_j, E_j)=G(C_1 \cup \cdots \cup C_j)$. Let p' equal the number of maximal paths of $G(C_j)$ that are contained in G_{j-1} (which equals the number of maximal paths of $G(C_j)$ that are not contained in G_{j-1}). Then

$$|E_j|-|V_j|=|E_{j-1}|-|V_{j-1}|+p'. \tag{2.1}$$

Let c equal the number of components of G_{j-1} and let p'' equal the number of components of G_{j-1} that C_j intersects. Then G_j has $c-p''+1$ components.

For a graph $G'=(V', E')$, let $\nu(G')$ denote the dimension of the cycle space of G'. It is well known that $\nu(G')=|E'|-|V'|+c'$ where c' is the number of components of G'.

Since G_{j-1} is a proper subgraph of G_j, and G_j is 2 edge-connected, $\nu(G_{j-1})<\nu(G_j)$. It follows that

$$\nu(G_j)=\nu(G_{j-1})+1$$

or

$$|E_j|-|V_j|+(c-p''+1)=|E_{j-1}|-|V_{j-1}|+c+1$$

or (substituting from (2.1))

$$|E_{j-1}|-|V_{j-1}|+p'+(c-p''+1)=|E_{j-1}|-|V_{j-1}|+c+1$$

or $p'=p''$. The result follows. □

An immediate corollary is the following characterization of ED cycle bases.

Corollary 2.6. *$\mathcal{P}$ is an ED cycle basis relative to the ordering $C_1, \ldots, C_d$ iff $G(C_j \backslash \{C_1 \cup \cdots \cup C_{j-1}\})$, for $2 \leqslant j \leqslant d$, is a path with exactly its endnodes in common with $G(C_1 \cup \cdots \cup C_{j-1})$.*

To understand our choice of the term ED cycle basis, recall the following definition (see [5]). A *nonseparable ear decomposition* for a graph G is a collection of subgraphs $P_0, \ldots, P_n$ which satisfy the following:

(2.2) P_0 is a cycle;

(2.3) P_{i+1} $(i \geqslant 0)$ is a path which has exactly its endnodes (called *attachment nodes*) in common with $P_0 \cup \cdots \cup P_i$;

(2.4) $G = P_0 \cup \cdots \cup P_n$.

Hence, if $C_1, \ldots, C_d$ is an ED cycle basis for a 2-connected graph G, then the graphs $G(C_j \backslash \{C_1 \cup \cdots \cup C_{j-1}\})$ $(1 \leqslant j \leqslant d)$ constitute a nonseparable ear decomposition for G. Conversely, it is easy to see that any nonseparable ear decomposition specifies an ED cycle basis.

3. Relationships of the cycle bases

In this section we show that our cycle bases satisfy the relationships shown in Fig. 1.

Proposition 3.1. *The SF cycle bases are properly contained in the ED cycle bases.*

Proof. Let $\mathcal{P}$ be a SF cycle basis. By Proposition 2.4, $\mathcal{P}$ has a block enlarging ordering. Hence $\mathcal{P}$ is an ED cycle basis.

To see that the containment is proper, consider G_1 in Fig. 2 and the cycle basis $\mathcal{P}_3 = \{\{e_1, e_3, e_5\}, \{e_2, e_4, e_5\}, \{e_3, e_4\}\}$. $\square$

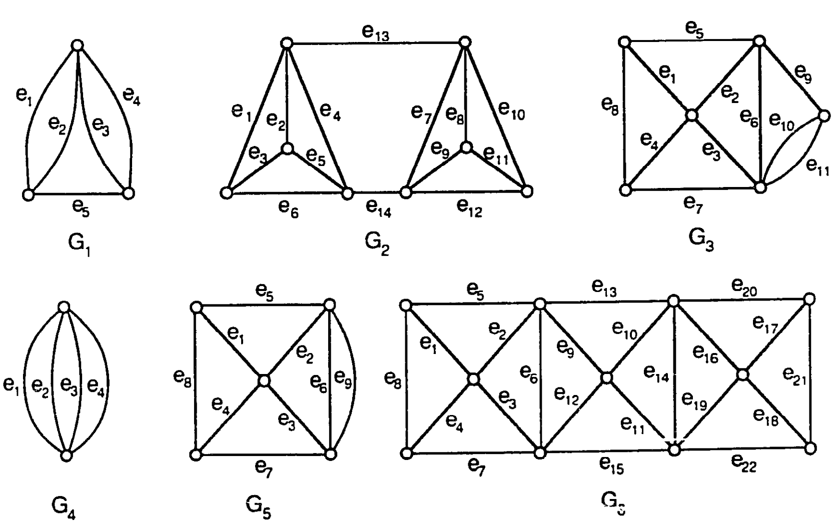


Fig. 2.

Observation 3.2. *$\mathscr{P}_3$ is ED, not SF and not SBE.*

Proposition 3.3. *The ED cycle bases are properly contained in the F cycle bases.*

Proof. We have containment by definition. To see that containment is proper, consider G_2 in Fig. 2 and the following cycle bases $\mathscr{P}_2$:

$$C_1 = \{e_1, e_4, e_6\}, \quad C_2 = \{e_2, e_4, e_5\}, \quad C_3 = \{e_1, e_4, e_5, e_3\},$$
$$C_4 = \{e_{10}, e_7, e_{12}\}, \quad C_5 = \{e_8, e_7, e_9\}, \quad C_6 = \{e_{10}, e_7, e_9, e_{11}\},$$
$$C_7 = \{e_2, e_3, e_6, e_{14}, e_{12}, e_{11}, e_8, e_{13}\}.$$

$\mathscr{P}_2$ is an F cycle basis relative to the given indexing. e_{13} and e_{14} are the only edges that occur in exactly one cycle of $\mathscr{P}_2$, both occurring in C_7. Hence, C_7 must be last in any fundamental ordering and $\mathscr{P}_2$ is not an ED cycle basis because $G(C_1 \cup \cdots \cup C_6))$ has one more block than G_2. □

Let $b(G)$ denote the number of blocks of G. Then by generalizing the construction of G_2 we can prove the following.

Corollary 3.4. *For any positive integer k, there exists a graph G which has an F cycle basis with ordering $C_1, \ldots, C_d$ such that $b(G(C_1 \cup \cdots \cup C_{d-1})) = b(G) + k$.*

Proposition 3.5. *The F cycle bases are properly contained in the BE cycle bases.*

Proof. Containment follows immediately from Proposition 2.4. To see that containment is proper, consider G_3 in Fig. 2 and the following cycle basis, $\mathscr{P}_1$, for G_3:

$$C_1 = \{e_1, e_2, e_6, e_7, e_8\}, \quad C_2 = \{e_2, e_3, e_7, e_8, e_5\},$$
$$C_3 = \{e_3, e_4, e_8, e_5, e_6\}, \quad C_4 = \{e_4, e_{11}, e_5, e_6, e_7\},$$
$$C_5 = \{e_9, e_{10}, e_6\}, \quad C_6 = \{e_{10}, e_{11}\}.$$

C_1, C_2, C_3 are C_4 are a 2-C cycle basis for $G(C_1 \cup \cdots \cup C_4)$. Hence $\mathscr{P}_1$ is not fundamental. □

Observation 3.6. *$\mathscr{P}_1$ is not F, not SBE and not 2-C.*

Observation 3.7. *The 2-C cycle bases are not F.*

Proposition 3.8. *There is no SBE cycle basis that is F but not ED.*

Proof. Let $\mathscr{P}$ be a cycle basis that is F but not ED. Then $\mathscr{P}$ has at least one fundamental ordering, yet, by definition, no such ordering is BE. □

With the following proposition we establish the remaining relationships depicted in Fig. 1.

Proposition 3.9. *There exist cycle bases that are:*

(3.1) *SBE and SF;*

(3.2) *SBE, ED and not SF;*

(3.3) *SF and not SBE;*

(3.4) *SBE and 2-C;*

(3.5) *SBE, not F and not 2-C;*

(3.6) *k-C and not SBE, for all $k \geq 2$.*

Proof. (3.1) Let $\mathcal{P}_5$ consist of a single cycle.

(3.2) For G_4 in Fig. 2, let $\mathcal{P}_6$ consist of $\{e_1, e_2\}$, $\{e_2, e_3\}$ and $\{e_3, e_4\}$.

(3.3) For G_1 in Fig. 2, let $\mathcal{P}_4$ consist of $\{e_1, e_2\}$, $\{e_3, e_4\}$ and $\{e_1, e_4, e_5\}$.

(3.4) For the graph induced by $e_1, \ldots, e_8$ of G_3 in Fig. 2, let $\mathcal{P}_8$ consist of C_1, C_2, C_3 and C_4 from $\mathcal{P}_1$.

(3.5) For G_5 in Fig. 3.2, let $\mathcal{P}_7$ consist of C_1, C_2, C_3 and C_4 from $\mathcal{P}_1$ plus $\{e_5, e_9, e_7, e_8\}$.

(3.6) For G_6 in Fig. 3.2, let $\mathcal{P}_9$ consist of $C_1, \ldots, C_4$ from $\mathcal{P}_1$ plus the following similar eight cycles:

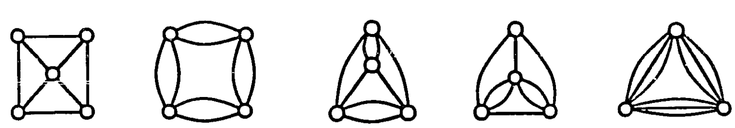
$$\{e_9, e_{10}, e_{14}, e_{15}, e_6\}, \quad \{e_{10}, e_{11}, e_{15}, e_6, e_{13}\},$$
$$\{e_{11}, e_{12}, e_6, e_{13}, e_{14}\}, \quad \{e_{12}, e_9, e_{13}, e_{14}, e_{15}\},$$
$$\{e_{16}, e_{17}, e_{21}, e_{22}, e_{14}\}, \quad \{e_{17}, e_{18}, e_{22}, e_{14}, e_{20}\},$$
$$\{e_{18}, e_{19}, e_{14}, e_{20}, e_{21}\}, \quad \{e_{19}, e_{16}, e_{20}, e_{21}, e_{22}\}.$$

This is the case $k = 2$. This cycle basis can be generalized easily for $k \geq 3$ by similarly using three copies of the graphs defined in Section 7. □

4. Characterizing the graphs for which every cycle basis is *F*

The definitions and results from this section appear explicitly and implicitly in [2]. We present them here for comparison with the results in the next two sections and because they are used in statements and a proof of results in the next section. Let $M_1, \ldots, M_5$ denote the graphs in Fig. 3.

Let $G = (V, E)$ be a graph and let $e = (u, v) \in E$. *Deleting* e is the operation which results in the graph $G' = (V, E \backslash e)$. *Contracting* e is the operation of



M_1 M_2 M_3 M_4 M_5

Fig. 3.

deleting e and identifying u and v. A *minor of* G is a graph which can be obtained from G by a sequence of deletions and contractions (and the removal of isolated nodes).

Theorem 4.1. *The following are equivalent for any 2-connected graph* G:

(4.1) *Every cycle basis of* G *is* F.
(4.2) G *has no* $M_1, \ldots, M_5$ *minor.*
(4.3) G *is a minor of a fence.*
(4.4) G *has no* 2-C *cycle basis.*

We observe that the fences are planar since M_1 is a minor of K_5 and $K_{3,3}$.

Let us next define the fences referred to above. To do this let us call *subdividing an edge* $e=(u, v)$ the operation of deleting e and replacing it with a path (of arbitrary positive length) connecting u to v.

Let $T_1=(V_1, E_1)$ and $T_2=(V_2, E_2)$ be isomorphic trees where $V_1 \cap V_2=\emptyset$, $E_1 \cap E_2=\emptyset$ and the isomorphism is determined by $\Theta: V_1 \rightarrow V_2$. Let $\tilde{G}=(\tilde{V}, \tilde{E})$ be defined by

$$\tilde{V}=V_1 \cup V_2 \qquad \tilde{E}=E_1 \cup E_2 \cup E_3$$

where

$$E_3=\{(v_1, v_2): v_1 \in V_1, v_2 \in V_2, \Theta(v_1)=v_2\}$$

A graph F obtained from $\tilde{G}$ by subdividing a subset of edges in $\tilde{E}$ is called a *fence without a gate.*

Let $\tilde{G}$ be defined as above and let $e=(u_1, u_2) \in E_3$. Consider the graphs G_1 and G_2 in Fig. 4. Let $\bar{G}$ be obtained from $\tilde{G}$ be deleting e and identifying w_1 with u_1 and w_2 with u_2 (or x_1 with u_1 and x_2 with u_2). Then the graph F obtained from $\bar{G}$ by subdividing a subset of edges of $\bar{G}$ is called a *fence with a gate.* The subgraph of F which is a subdivision of G_1 (respectively, G_2) is called the *gate.* We note that, by definition, a fence with a gate has exactly one gate.

Fig. 5 contains an example of a fence with a G_1 gate. In accordance with this way of drawing fences we make the following definitions: Let F be an arbitrary fence and let $\tilde{G}$ be the associated underlying graph as defined above. Any path of F which is the result of subdividing an edge in $E_1 \cup E_2$ is called a *horizontal path* and any path which is the result of subdividing an edge in E_3 is called a *vertical*

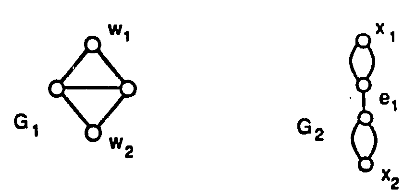


Fig. 4.

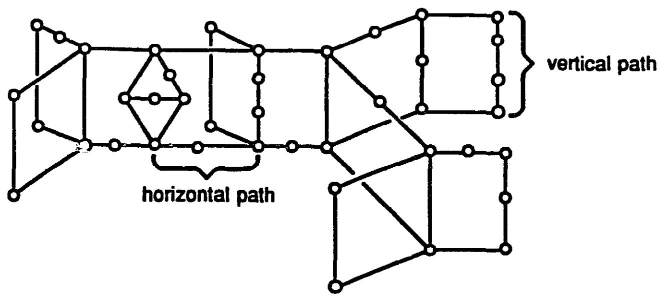


Fig. 5.

path. (These paths may have length 1.) The edges in a horizontal (vertical) path are called *horizontal* (*vertical*) *edges*. (The edges in a gate are neither horizontal nor vertical.)

A *face* of a fence F is a connected subgraph of F whose edges can be partitioned into the edge sets of exactly two horizontal paths and either two vertical paths or one vertical path and the gate. An *extreme vertical path* is a vertical path contained in exactly one face.

In the course of proving Theorem 4.1, the following two results are proved.

Theorem 4.2. *If a 2-connected graph G is a minor of a fence, then G can be obtained from a fence F by applications of the following operations*:

(4.5) *Contracting horizontal edges of F.*

(4.6) *Contracting the edge e_1 (see Fig. 4) if F contains a gate G_2 (and e_1 has not been subdivided).*

(4.7) *Deleting vertical edges of F.*

Proposition 4.3. *Let G be a graph obtained from a fence F by applications of operations* (4.5), (4.6) *and* (4.7). *Then, for every cycle basis $\mathcal{P}$ for G, there exists a path P in G which satisfies the following*:

(4.8) *The edges in P are either the edges of a non-extreme vertical path in F or the edges of an extreme vertical path plus the two adjacent horizontal paths in F.*

(4.9) *The edges in P are covered exactly once by $\mathcal{P}$.*

(This result follows from the proof of Claim 2 of Theorem 6.2 in [2].)

5. Characterizing the graphs for which every cycle basis is ED

In this section we show that the class of graphs for which every cycle basis is ED is the same as the class of graphs for which every cycle basis is F. In particular, we show that if every cycle basis for a graph is F, then every cycle basis has a fundamental ordering that is block expanding (although not every fundamental ordering need be block expanding). The main result is the following.

Theorem 5.1. *The following are equivalent for any 2-connected graph G:*
(5.1) *Every cycle basis of G is ED.*
(5.2) *G has no $M_1, \ldots, M_5$ minor.*
(5.3) *G is a minor of a fence.*
(5.4) *G has no 2-C cycle basis.*

Corollary 5.2. *A 2-connected graph has a cycle basis that is not ED iff it has a 2-C cycle basis.*

Proof of Theorem 5.1. By Theorem 4.1, (5.2), (5.3) and (5.4) are equivalent. By definition, the ED cycle bases are F. Hence, it suffices to show that (5.3)$\Rightarrow$(5.1).

By Theorem 4.2. G can be obtained from a fence $F = (V, E)$ by applications of operations (4.5), (4.6) and (4.7). Let X, $Y \subseteq E$ be the sets of horizontal and vertical edges in F, respectively, which are contracted and deleted, respectively, to produce G. Let F' be the graph obtained from F by deleting the edges in Y.

Let m denote the number of vertical paths of F that are not deleted in producing F'. We prove the result by induction on m. Assume that if F has no gate, then $m \geq 3$ and if F has a gate, then $m \geq 2$. Otherwise the result follows.

Let $\mathscr{P}$ be a cycle basis for G. By Proposition 4.3, there exists a path P in G which satisfies (4.8) and (4.9). Let C be the unique cycle in $\mathscr{P}$ that covers the edges in P. Suppose there exists an edge e, not in P, whose edges are covered exactly once by $\mathscr{P}$, and C is the cycle in $\mathscr{P}$ that covers e. By (4.8) and our lower bounds on m, the endnodes of P have degree ≥ 3 in G. Hence, G with P deleted is 2-connected. But, by Proposition 2.5, deleting the edges in P from G results in a graph G' in which e is a cut edge. Thus, there is no such edge e. If we delete P from G to get G', then $\mathscr{P} \backslash C$ is a cycle basis for G'. Let F'' be the fence obtained from F by deleting P. G is a minor of F''. We can now apply the inductive hypothesis. □

Let us observe that if every cycle basis of a graph G is F, then G may have fundamental orderings that are not block expanding. Consider the graph G in Fig. 2, which is a minor of a fence, and the cycle basis described in the proof of (3.3).

6. Characterizations of the graphs for which every cycle basis is SF and SBE, respectively

In this section we present simple characterizations of those graphs for which every cycle basis is SF and SBE, respectively. The first class of graphs is properly contained in the fences. The second class intersects the fences but also contains nonplanar graphs.

Let K_4, M_6, M_7, M_7', and M_8 denote the graphs in Fig. 6. A graph H' is *homeomorphic* to a graph H if there exists a graph H'' such that H'' is isomorphic

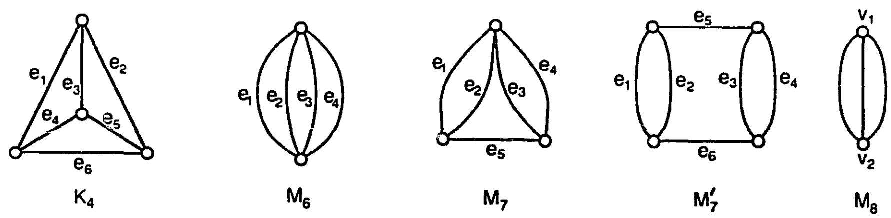


Fig. 6.

to a subdivision of H and isomorphic to a subdivision of H'. A *bond* is a graph on two nodes that consists of any number of edges connecting the two nodes.

Our proofs make use of the following result (see [5]).

Proposition 6.1. *A graph is 2-connected iff it has a nonseparable ear decomposition.*

Theorem 6.2. *The following are equivalent for any 2-connected graph G:*

(6.1) *Every cycle basis of G is SF.*

(6.2) *G has no K_4 or M_6 minor.*

(6.3) *G is a subdivision of a bond with two or three edges.*

(6.4) *For every cycle C of G, there is no cycle basis for G that covers every edge of C two or more times.*

Proof. (6.3)$\Rightarrow$(6.4) and (6.4)$\Rightarrow$(6.1) are obvious.

(6.2)$\Rightarrow$(6.3) Let $P_0, P_1, \ldots, P_n$ be a nonseparable ear decomposition for G. If $n=0$ or 1, the result follows immediately, so let us assume $n \geq 2$. Consider the graph $G_1 = P_0 \cup P_1$. G_1 may be viewed as a subdivision of M_8 where $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ denote the three paths from v_1 to v_2 (see Fig. 6). Let u_1 and u_2 be the attachment nodes of P_2. If u_1 and u_2 are on the same path $\bar{P}_i$, then $G_2 = P_0 \cup P_1 \cup P_2$, hence G, contains an M_6 minor. Otherwise u_1 and u_2 are interior nodes on paths $\bar{P}_i$ and $\bar{P}_j$ where $i \neq j$. In this case G_2, hence G, contains a K_4 minor. So, $G = G_1$.

(6.1)$\Rightarrow$(6.2) Suppose G has a K_4 minor. Since all nodes of K_4 have degree 3, a graph has a K_4 minor iff it has a subgraph homeomorphic to K_4. Let $\bar{K}_4$ be a subgraph of G which is homeomorphic to K_4.

The following list (see Fig. 6) describes a cycle basis for K_4 that is not SF: $\{e_1, e_2, e_4, e_5\}$, $\{e_4, e_5, e_6\}$ and $\{e_1, e_3, e_5, e_6\}$. This cycle basis induces a cycle basis $\mathcal{P}'$ for $\bar{K}_4$. And any cycle basis for G that contains $\mathcal{P}'$ is not SF.

Suppose G has no K_4 minor, but has an M_6 minor. Let $P_0, \ldots, P_n$ be a nonseparable ear decomposition for G. To have an M_6 minor, $n \geq 2$. $G_2 = P_0 \cup P_1 \cup P_2$ can be obtained from M_6, M_7 or M'_7 in Fig. 6 by subdividing. (Otherwise G has a K_4 minor as in the proof of (6.2)$\Rightarrow$(6.3).)

The following cycles (see Fig. 6) define cycle bases $\mathcal{P}$, $\mathcal{P}'$, and $\mathcal{P}''$ for M_6, M_7

and M_7', respectively, that are not SF:

$$\mathcal{P} = \{\{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_4\}\},$$
$$\mathcal{P}' = \{\{e_1, e_3, e_5\}, \{e_2, e_4, e_5\}, \{e_3, e_4\}\} \quad \text{and}$$
$$\mathcal{P}'' = \{\{e_1, e_5, e_3, e_6\}, \{e_2, e_5, e_4, e_6\}, \{e_3, e_4\}\}.$$

Each of these then induces a cycle basis for G which is not SF. □

Theorem 6.3. *The following are equivalent for any 2-connected graph G:*
(6.5) *Every cycle basis is SBE.*
(6.6) *G has no subgraph homeomorphic to M_7 or M_7'.*
(6.7) *G is a subdivision of either K_4, $K_{3,3}$ or a bond.*
(6.8) *Every two cycles of G have two or more common nodes.*

Proof. (6.5)$\Rightarrow$(6.6), (6.7)$\Rightarrow$(6.8), and (6.8)$\Rightarrow$(6.5) are obvious.

(6.6)$\Rightarrow$(6.7) Let $P_0, P_1, \ldots, P_n$ be a nonseparable ear decomposition for G. If $n = 0$ or 1, the result follows immediately. Suppose $n = 2$. If $G_1 = P_0 \cup P_1$, then G_1 may be viewed as a subdivision of M_8 where $\bar{P}_1$, $\bar{P}_2$ and $\bar{P}_3$ denote the three paths from v_1 to v_2 (see Fig. 6). Let u_1 and u_2 be the attachment nodes of P_2. Suppose $\{u_1, u_2\}$ is different from $\{v_1, v_2\}$. If u_1 and u_2 are on the same path $\bar{P}_i$ then $G_2 = P_0 \cup P_1 \cup P_2$ contains a subgraph homeomorphic to M_7 or M_7'. Otherwise, u_1 and u_2 are interior nodes on paths $\bar{P}_i$ and $\bar{P}_j$ where $i \neq j$. In this case G_2 is a subdivision of K_4.

If $n = 2$ we are done. Suppose $n \geqslant 3$. Let w_1 and w_2 be the attachment nodes of P_3 on G_2. It is easy to check that in only one case adding P_3 does not result in a graph with an M_7 or M_7' homeomorph. In this case w_1 and w_2 are interior nodes on two different paths from $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ and P_3 such that the two paths share no endnode. Such a G_3 is a subdivision of $K_{3,3}$. Any additional ear results in an M_7 or M_7' minor.

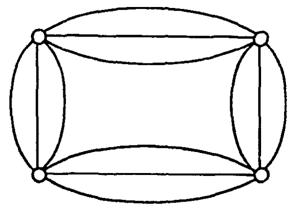
The remaining case is that u_1 and u_2 for P_2 are v_1 and v_2. In this case $G_2 = P_0 \cup P_1 \cup P_2$ is a subdivision of a bond. By similar reasoning, all subsequent ears must also attach at v_1 and v_2, hence G is a subdivision of a bond. □

7. Existence of k-C cycles bases

In this section we exhibit, for every positive integer k, a graph G_k that has a k-C cycle basis. In fact, we show the following somewhat stronger result.

Theorem 7.1. *For every positive odd integer k, there exists a graph G_k with a k-C cycle basis that covers every edge of G_k exactly k times.*

Proof. Let $G_k = (V, E)$ be the graph obtained from a cycle of length $k+1$ by replacing each edge with k parallel edges. Consider the 'natural' planar embedding of G_k, as illustrated for the case $k = 3$ in Fig. 7.



G_3

Fig. 7.

Let us partition the edges in G_k into k cycles by applying the following algorithm to the planar embedding.

Algorithm

For $i = 1$ to k, **do**

(1) Let C_i = edges on outer face.

(2) Delete edges in C_i

End.

We construct the k-C cycle basis $\mathscr{P}$ as follows: For $1 \leqslant i \leqslant k-1$, let $\mathscr{P}_i$ contain the $k+1$ cycles formed by removing, in turn, one edge of C_i and replacing it with the unique edge in C_k that forms a cycle. Let $\mathscr{P} = \cup \{\mathscr{P}_i\colon 1 \leqslant i \leqslant k-1\} \cup \{C_k\}$.

The dimension of the cycle space of $G_k = (V, E)$ is $|E| - |V| + 1 = k(k+1) - (k+1) + 1 = k^2 = |\mathscr{P}|$. To show that the cycles in $\mathscr{P}$ are independent, we show that every cycle in a cycle basis $\mathscr{P}'$ for G_k can be expressed as a sum of cycles in $\mathscr{P}$.

We take $\mathscr{P}'$ to be all cycles of length two that contain an edge in C_k plus the cycle C_k. Again $|\mathscr{P}'| = k^2$. It is easy to see that every interior face of G_k can be expressed as a sum of (at most two) cycles in $\mathscr{P}'$. It is well known that the interior faces of a plane graph constitute a cycle basis. Hence, $\mathscr{P}'$ is a cycle basis for G_k.

Finally, every cycle in $\mathscr{P}'$ can be expressed as a sum of cycles in $\mathscr{P}$ as follows: Let $C = \{e_i, e_k\}$ be a cycle of $\mathscr{P}'$ where $e_i \in C_i$, $1 \leqslant i \leqslant k-1$, and $e_k \in C_k$. Then, since k is odd, C is the sum of the k cycles in $\mathscr{P}_i$ that do not contain e_k, plus C_k. $\square$

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